Exercise 6D: Solutions

1 a Converse: If $x = 1$, then $2x + 3 = 5$.
Proof: If $x = 1$ then

$$2x + 3 = 2 \times 1 + 3 = 5.$$

b Converse: If $n - 3$ is even, then n is odd.
Proof: If $n - 3$ is even then $n - 3 = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$n = 2k + 3 = 2k + 2 + 1 = 2(k + 1) + 1$$

is odd.

c Converse: If m is odd, then $m^2 + 2m + 1$ is even.
Proof 1: If m is odd then the expression $m^2 + 2m + 1$ is of the form,

$$\text{odd} \; + \; \text{even} \; + \; \text{odd} \; = \; \text{even}.$$

Proof 2: If m is odd then $m = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m^2 + 2m + 1 &= (2k + 1)^2 + 2(k + 1) + 1 \\ &= 4k^2 + 4k + 1 + 2k + 2 + 1 \\ &= 4k^2 + 6k + 3 \\ &= 4k^2 + 6k + 2 + 1 \\ &= 2(2k^2 + 3k + 1) + 1, \end{aligned}$$

is clearly odd.

d Converse: If n is divisible by 5, then n^2 is divisible by 5.
Proof: If n is divisible by 5 then $n = 5k$ for some $k \in \mathbb{Z}$. Therefore,

$$n^2 = (5k)^2 = 25k^2 = 5(5k^2),$$

which is divisible by 5.

2 a Converse: If mn is a multiple of 4, then m and n are even.

b This statement is not true. For instance, 4×1 is a multiple of 4, and yet 1 is clearly not even.

3 a These statements are not equivalent. $(P \Rightarrow Q)$
If Vivian is in China then she is in Asia, since Asia is a country in China.
$(Q \nRightarrow P)$ If Vivian is in Asia, she is not necessarily in China. For example, she could be in Japan.

b These statements are equivalent.
$(P \Rightarrow Q)$ If $2x = 4$, then dividing both sides by 2 gives $x = 2$.
$(Q \Rightarrow P)$ If $x = 2$, then multiplying both sides by 2 gives $2x = 4$.

c These statements are not equivalent.
$(P \Rightarrow Q)$ If $x > 0$ and $y > 0$ then $xy > 0$ since the product of two positive numbers is positive.
$(Q \nRightarrow P)$ If $xy > 0$, then it may not be true that $x > 0$ and $y > 0$. For example, $(-1) \times (-1) > 0$, however $-1 < 0$.

d These statements are equivalent.
$(P \Rightarrow Q)$ If m or n are even then mn will be even.
$(Q \Rightarrow P)$ If mn is even then either m or n are even since otherwise the product of two odds numbers would give an odd number.

4 $(\Rightarrow)$ If $n + 1$ is odd then, $n + 1 = 2k + 1$, where $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n + 2 &= 2k + 2 \\ &= 2(k + 1), \end{aligned}$$

so that $n + 2$ is even.
$(\Leftarrow)$ If $n + 2$ is even then, $n + 2 = 2k$, where $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n+1 &= 2k-1 \\ &= 2k-2+1 \\ &= 2(k-1)+1 \end{aligned}$$

so that $n+1$ is odd.

5 ($\Rightarrow$) Suppose that n^2-4 is prime. Since

$$n^2-4=(n-2)(n+2)$$

expresses n^2-4 as the product of two numbers, either $n-2=1$ or $n+2=1$. Therefore, $n=3$ or $n=-1$. However, n must be positive, so $n=3$.

($\Leftarrow$) If $n=3$ then

$$n^2-4=3^2-4=5$$

is prime.

6 ($\Rightarrow$) We prove this statement in the contrapositive. Suppose n is not even. Then $n=2k+1$ where $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3+12k^2+6k+1 \\ &= 2(4k^4+6k^2+3k)+1 \end{aligned}$$

is odd.

($\Leftarrow$) If n is even then $n=2k$. Therefore,

$$\begin{aligned} n^3 &= (2k)^3 \\ &= 8k^3 \\ &= 2(4k^3) \end{aligned}$$

is even.

7 ($\Rightarrow$) Suppose that n is odd. Then $n=2m+1$, for some $m \in \mathbb{Z}$. Now either m is even or m is odd. If m is even, then $m=2k$ so that

$$\begin{aligned} n &= 2m+1 \\ &= 2(2k)+1 \\ &= 4k+1. \end{aligned}$$

as required. If m is odd then $m=2q+1$ so that

$$\begin{aligned} n &= 2m+1 \\ &= 2(2q+1)+1 \\ &= 4q+3 \\ &= 4q+4-1 \\ &= 4(q+1)-1 \\ &= 4k-1, \text{ where } k=q+1, \end{aligned}$$

as required.

($\Leftarrow$) If $n=4k\pm1$ then either $n=4k+1$ or $n=4k-1$. If $n=4k+1$, then

$$\begin{aligned} n &= 4k+1 \\ &= 2(2k)+1 \\ &= 2m+1, \text{ where } m=2k, \end{aligned}$$

is odd, as required. Likewise, if $n=4k-1$, then

$$\begin{aligned} n &= 4k - 1 \\ &= 4k - 2 + 1 \\ &= 2(2k-1) + 1 \\ &= 2m + 1, \text{ where } m = 2k - 1, \end{aligned}$$

is odd, as required.

8 (⇒) Suppose that,

$$\begin{aligned} (x+y)^2 &= x^2 + y^2 \\ x^2 + 2xy + y^2 &= x^2 + y^2 \\ 2xy &= 0 \\ xy &= 0 \end{aligned}$$

Therefore, $x = 0$ or $y = 0$.
(⇐) Suppose that $x = 0$ or $y = 0$. We can assume that $x = 0$. Then

$$\begin{aligned} (x+y)^2 &= (0+y)^2 \\ &= y^2 \\ &= 0^2 + y^2 \\ &= x^2 + y^2, \end{aligned}$$

as required.

9 a Expanding gives

$$\begin{aligned} (m-n)(m^2+mn+n^2) &= m^3 + m^2n + mn^2 - m^2n - mn^2 - n^3 \\ &= m^3 - n^3. \end{aligned}$$

b (⇐) We will prove this in the contrapositive. Suppose that $m - n$ were odd. Then either m is odd and n is even or visa versa.
Case 1 - If m is odd and n is even
The expression $m^2 + mn + n^2$ is of the form,

$$\text{odd} + \text{even} + \text{even} = \text{odd}.$$

Case 2 - m is even and n is odd
The expression $m^2 + mn + n^2$ is of the form,

$$\text{even} + \text{even} + \text{odd} = \text{odd}.$$

In both instances, the expression $m^2 + mn + n^2$ is odd. Therefore, $m^3 - n^3 = (m-n)(m^2+mn+n^2)$ is the product of two odd numbers, and will therefore be odd.

10 We first note that any integer n can be written in the form $n = 100x + y$ where $x, y \in \mathbb{Z}$ and y is the number formed by the last two digits. For example, $1234 = 100 \times 12 + 34$. Then

$$\begin{aligned} & n \text{ is divisible by } 4 \\ \Leftrightarrow & n = 100x + y = 4k, \text{for some } k \in \mathbb{Z} \\ \Leftrightarrow & y = 4k - 100x \\ \Leftrightarrow & y = 4(k - 25x) \\ \Leftrightarrow & y \text{ is divisible by } 4. \end{aligned}$$